The Brauer Group through the Lens of Crossed Product Algebras

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What is a division algebra over a field F?

finite dimensional.

Definition

Given a field F, a **division algebra over F** is an associative, unital algebra over F (i.e. vector space over F with multiplication such that $1_A F = Z(A)$) and such that all nonzero elements are invertible.

Ex: 1) F is division alg over F.

2) $\mathbb{H}$ division alg over $\mathbb{R}$.

$\leadsto \{1, i, j, k\}$

$$i^2 = j^2 = k^2 = -1$$

$$k = ij = -ji$$

Classification

$\dim_{\mathbb{C}} D = n.$

What are all of the division algebras over $\mathbb{C}$? $\longrightarrow$ $\mathbb{C}$

Suppose D division over $\mathbb{C}$.

$d \in D \setminus \mathbb{C}$

$\mathbb{C}(d)$ <u>field</u>

linearly dep. $/\mathbb{C}$ $\quad 1 \quad d \quad d^2 \quad d^3 \; \ldots \ldots \; d^n$

d is algebraic over $\mathbb{C}$

$\mathbb{C}(d) = \mathbb{C}$ b/c alg. closed.

$\mathbb{H} \otimes_{\mathbb{R}} \mathbb{H}$

What are all of the division algebras over $\mathbb{R}$?

$\mathbb{R}$, $\mathbb{H}$

what happens when you take tensor product of two division algebras over F?

Not necessarily division.

$$\mathbb{H} \otimes_{\mathbb{R}} \mathbb{H} \cong \underbrace{M_4(\mathbb{R})}_{\text{CSA}}$$

that's not division

Central Simple Algebras over a field F

Definition

Given a field F, a **central simple algebra (CSA) over** F is an associative, unital, finite dimensional algebra, A, over F such that:

(1) A is central ($Z(A) = 1_A F$)

(2) A is simple (i.e. A has no proper, nontrivial, two-sided ideals.)

CSA's

1) Tensor of two CSA's is a CSA. (over the same field).

2) Every division alg is a CSA over its center.

3) Every CSA$/F$ is of the form $M_n(D)$ for some Division alg D/F (Wedderburn's Thm).

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What is the Brauer group?

Definition

The Brauer Group of a field F, denoted $Br(F)$, is the set of all CSA's over F modulo the equivalence relation $\sim$, where

$$A \sim B \iff A \otimes_F M_{n_1}(F) \cong B \otimes_F M_{n_2}(F)$$

for some $n_1, n_2 \in \mathbb{Z}_{>0}$.

Equivalence Relation?

$A \cong M_n(D) \cong D \otimes M_n(F)$

$[A] = [D]$.

We have $A \sim B \iff A \otimes_F M_{n_1}(F) \cong B \otimes_F M_{n_2}(F)$ for some $n_1, n_2 \in \mathbb{Z}_{>0}$.

$A \otimes_F M_n(F) \cong B \otimes_F M_m(F)$

Kronecker product.

$B \otimes_F M_\ell(F) \cong C \otimes_F M_k(F)$

$A \otimes_F \underbrace{M_{n\ell}(F)} \cong A \otimes_F (\underbrace{M_n(F) \otimes M_\ell(F)})$

$\cong (A \otimes_F M_n(F)) \otimes M_\ell(F)$

$\cong (B \otimes_F M_m(F)) \otimes M_\ell(F)$

$\vdots$

$\cong C \otimes M_{km}(F)$

$A \sim C$.

How is this a group?

$\{ \text{CSA's over } F \} / \sim \, , \otimes$ (I promise $\otimes$ is well defined).

- associative ✓

- what's the identity elt.

$$[F] = [M_n(F)].$$

$$[A] \otimes [M_n(F)] = [A \otimes M_n(F)] = A. \quad ✓$$

- what is the inverse of $[A]$?

$$A^{op} = \{ a^o \mid a \in A \}$$

$$a^o b^o = (ba)^o$$

The Opposite Algebra as the Inverse

$n = \dim_F A$

WTS: $[A] \otimes [A^{op}] = [M_n(F)]$. where

$$\varphi : A \otimes_F A^{op} \xrightarrow{\sim} \mathrm{End}_F(A) \cong M_n(F)$$

sandwich

$$a \otimes b^\circ \longmapsto (\underbrace{x \longmapsto axb})$$

$$\varphi(\underbrace{(a \otimes b^\circ)(c \otimes d^\circ)}) = \underbrace{\varphi(a \otimes b^\circ)}\,\underbrace{\varphi(c \otimes d^\circ)}$$

$ac \otimes b^\circ d^\circ$

$ac \otimes (db)^\circ$

$$\underbrace{a\,\underbrace{c\,x\,d}\,b}.$$

Extra Space

$A \otimes A^{op}$ is CSA / F

- $\ker \varphi \neq A \otimes A^{op} \Rightarrow \ker \varphi = 0$
  so φ is injective.

- $\dim_F A = n$

$\dim_F A\otimes A^{op} = n^2 = \dim \mathrm{End}_F A$. $\square$.

$Br(F)$ is an abelian gp.

- $Br(\mathbb{C})$ is trivial.

Some Remarks about the Brauer Group

$$\mathbb{H} \otimes_{\mathbb{R}} \mathbb{H} = M_4(\mathbb{R}).$$

$$\mathbb{H} \cong \mathbb{H}^{op}.$$

$$[\mathbb{H}]^2 = [\mathbb{R}]$$

$$\text{ord}\,[\mathbb{H}] = 2. \text{ in } Br(\mathbb{R}).$$

$$Br(\mathbb{R}) \cong \mathbb{Z}/2\mathbb{Z}.$$

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Let's build some CSA's

- E a finite Galois extension of F.

$$A = \bigoplus_{\sigma \in G} u_\sigma E$$

$\sigma \curvearrowright e \in E$

e^σ

$1_A E$

Need to define mult:

$$\left(\sum_{\sigma \in G} u_\sigma c_\sigma\right)\left(\sum_{\tau \in G} u_\tau d_\tau\right) = \sum_{\sigma,\tau \in G} u_{\sigma\tau} \Phi(\sigma,\tau) c_\sigma^\tau d_\tau$$

$$\Phi : G \times G \longrightarrow E^\times$$

What do we need $\Phi : G \times G \to E^\times$ to satisfy?

Want A to be associative!

$$(u_\sigma u_\tau)u_\rho = (u_{\sigma\tau}\Phi(\sigma,\tau))u_\rho$$

$$= u_{\sigma\tau\rho}\boxed{\Phi(\sigma\tau,\rho)\,\Phi(\sigma,\tau)^\rho}$$

should be equal to

$$u_\sigma(u_\tau u_\rho) = u_\sigma(u_{\tau\rho}\Phi(\tau,\rho))$$

$$= u_{\sigma\tau\rho}\boxed{\Phi(\sigma,\tau\rho)\,\Phi(\tau,\rho)}.$$

Φ should satisfy:

$$\Phi(\sigma\tau,\rho)\,\Phi(\sigma,\tau)^\rho = \Phi(\sigma,\tau\rho)\,\Phi(\tau,\rho).$$

Extra Space

$$\mathbb{1}_A = \mathcal{U}_{id}\, \Phi(id_G, id_G)^{-1}$$

Normalize : $\Phi(id_G, id_G) = 1$

Why is A central?

E/F

$$x = \sum_{\sigma \in G} u_\sigma c_\sigma \in Z(A)$$

$\forall d \in E$ $\quad (1_A d)x = x(1_A d)$

$$\Rightarrow \quad 0 = (1_A d)x - x(1_A d) = \sum_{\sigma \in G} u_\sigma d^\sigma c_\sigma - \sum_{\sigma \in G} u_\sigma c_\sigma d$$

$$= \sum_{\sigma \in G} u_d \underbrace{(d^\sigma - d) c_\sigma}$$

either $\rightarrow d = d^\sigma \ \forall \sigma \in G.$

$$\Rightarrow \quad x = u_{id_G} c = 1_A c' \quad \ldots$$

$c' \in F$

Extra Space

$$A = (E, G, \underset{\tau}{\underset{\uparrow}{\Phi}}) \cong (E, G, \Psi)$$

$$\iff \Phi(\sigma,\tau) = \underbrace{\theta(\tau)\,\theta(\sigma\tau)^{-1}\,\theta(\sigma)^{\tau}}\,\Psi(\sigma,\tau).$$

$$\theta : G \to E^{\times}$$

$$[(E, G, \Phi)] \longrightarrow H^2(G, E^{\times}).$$

$$Br(E/F). = \text{CSA's over } F \text{ s.t. } [A \otimes_F E] = [E].$$

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Main Connection

Theorem

If E/F is a finite Galois extension with $G = Gal(E/F)$, then the mapping

$$\Theta_{E/F} : [\Phi] \to [(E, G, \Phi)]$$

is an isomorphism of $H^2(G, E^\times)$ to $Br(E/F)$.